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ASYMPTOTIC REPRESENTATIONS OF THE SOLUTION OF TIMOSHENKO'S INTEGRAL EQUATION IN THE THEORY OF LATERAL IMPACT†

A. P. ZAIKA and N. V. SLONOVSKII

Khar'kov

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Asymptotic methods of non-linear mechanics are used to obtain representations of the solution of Timoshenko's equation for a ball striking a rod, over the whole spectrum of the rod's fundamental modes of vibration.

IN DIMENSIONLESS notation, Timoshenko's integral equation for the lateral impact of a ball with a rod [1] is

$$s_0(\tau)-\int_0^\tau (\tau-\tau_1)p(\tau_1)d\tau_1 - s_1 p^{2/3}(\tau) - s_2 L(p) = 0 \tag{1}$$

$$L(p)=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\int_0^\tau p(\tau_1)\sin[s_3(2n-1)(\tau-\tau_1)]d\tau_1$$

$$p=P/P_m^0,\quad \tau=t/t_1^0 \tag{2}$$

where P_m^0 and t_1^0 are the maximum force and duration of the impact according to Hertz's theory [2] (with the rod replaced by a semi-bounded body); the parameters $s_0,\ldots,s_3$ are uniquely defined by condition (2).

We have

$$d^2L(p)/d\tau^2 = L(p'')$$

$$L(p)=\frac{\pi^4}{96}\frac{1}{s_3}p(\tau)+\frac{1}{s_3^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}\times$$

$$\times\int_0^\tau p(\tau_1)\sin[s_3(2n-1)^2(\tau-\tau_1)]d\tau_1 \tag{3}$$

$$L(p)=\frac{1}{2}s_3\int_0^\tau p(\tau_1)(\tau-\tau_1)d\tau_1\int_0^1\theta_2[0|\frac{4}{\pi}s_3(\tau-\tau_1)\gamma]d\gamma$$

Primes denote differentiation with respect to τ.

The first equality in (3) is obtained by substituting $\tau_* \to \tau-\tau_1$ before differentiation, and the third one by substituting the series representation of the theta-function $\theta_2(0|\cdot)$ and then integrating term by term; τ_* is the new variable of integration.

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We consider the case

$$s_3 \gg 1 \tag{4}$$

(the rod has low flexibility and a high-frequency spectrum of natural modes of transverse vibrations).

From Eq. (1) we have $s_1 p^{3/2} \sim s_0 \tau,\ \tau \ll 1$, and hence

$$p''(\tau) \sim {}^3/_4 (s_0/s_1)^{2/3}\, \tau^{-1/3},\quad \tau \ll 1 \tag{5}$$

By (4), (5) and the first equality of (3)

$$\begin{gathered} L''_{\tau^2}(p) = {}^3/_4 \sqrt{\pi} (s_0/s_1)^{2/3}\, (s_3)^{-2/3} \times \\ \times \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos[(2n-1)^2 s_3 \tau + \pi/4],\quad s_3 \gg 1 \end{gathered} \tag{6}$$

The derivation uses the relationship

$$\int_0^{s_3 \tau} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \tau_1 \frac{d\tau_1}{\sqrt{\tau_1}} \sim \sqrt{\pi/2},\quad s_3 \gg 1$$

Differentiating Eq. (1) on the basis of (6) (in view of rapid convergence we have retained the first term of the series in (6)), we obtain Duffing's equation (with fundamental mode zero and without friction)

$$\begin{gathered} q''_{\tau^2}(\tau) + s_1^{-1} q^{3/2}(\tau) = E^* \cos(s_3 \tau + \pi/4) \\ q = p^{2/3},\quad E^* = {}^3/_4 \sqrt{\pi} \frac{s_0^{2/3}}{s_1^{5/3}} \frac{s_2}{s_3^{2/3}} \end{gathered} \tag{7}$$

The initial conditions are

$$q(0) \equiv q_0 = 0,\quad q'(0) \equiv q'_0 = s_0/s_1 \tag{8}$$

(the second condition follows from (5)).

It can be shown that the high-frequency asymptotic behaviour $(k \to \infty)$ of the solution of a Duffing-type equation

$$\begin{gathered} u'' + A_0 u^s(\tau) = G \cos(k\tau + \alpha) \\ A_1 < A_0 < A_2;\quad 0 < G < G_1;\quad A_1, A_2,\ G_1 > 0 \end{gathered} \tag{9}$$

$(k, \alpha, s$ are real numbers) with initial conditions

$$u(0) \equiv u_0 = 0,\quad u'(0) = u'_0 \tag{10}$$

is given by

$$[(s+1)\Omega]^{-1} [B_\mu(\frac{1}{s+1}, \frac{1}{2}) - B_{\mu_0}(\frac{1}{s+1}, \frac{1}{2}) = \tau \tag{11}$$

The distance from $\tau = 0$ to τ_1, the latter being the first zero (or the second one if $u_0 = 0$), and the maximum u_m of $u(\tau)$ $(0 \ll \tau \ll \tau_1)$ are given by

$$\tau_1 = [(s+1)\Omega]^{-1}\left[2B\left(\frac{1}{s+1}, \frac{1}{2}\right) - B_{\mu_0}\left(\frac{1}{s+1}, \frac{1}{2}\right)\right]$$

$$u_m = \left\{\frac{s+1}{2A_0}\left[u_0'^2 + \frac{2A_0}{s+1} u_0^{s+1} - 2G/k u_0' \cos\alpha\right]\right\}^{1/(s+1)} \tag{12}$$

$$\Omega = \sqrt{\frac{2A_0}{s+1}}\, u^{\frac{s-1}{2}}, \quad \mu = \left[\frac{u(\tau)}{u_m}\right]^{s+1}, \quad \mu_0 = \left|\frac{u_0}{u_m}\right|^{s+1}$$

Here $B(\cdot,\cdot)$, $B_\mu(\cdot,\cdot)$ are the beta-function and the incomplete beta-function.

It follows from (11) and (12) that the perturbation has no effect on $u(\tau)$ if one of the following conditions holds

$$k = \infty, \quad u_0' = 0, \quad \alpha = \pi/2 \tag{13}$$

Thus, the equality $g = 0$ is equivalent to relations (13). The values of u_m and τ_1 may be larger or smaller than the unperturbed ones, depending on the signs of u_0' and $\cos\alpha$ and whether $s < 1$ or $s > 1$. At $s = 1$ there is an isochronic effect—the "frequency" Ω is independent of u_0 and u_0'.

Relations (11) and (12) were obtained by multiplying Eq. (9) by $u'(\tau)$ and integrating from 0 to τ. Integrating by parts in the trigonometric integral, we obtain the following asymptotic relation, which holds as $k \to \infty$

$$u'^2(\tau) - 2Gk^{-1}\sin(\alpha + k\tau)u'(\tau) - M = 0$$

$$M = u_0'^2 - \frac{2A_0}{s+1}\left(u^{s+1} - u_0^{s+1}\right) - 2\frac{G}{k} u_0' \sin\alpha \tag{14}$$

The maximum u_m is defined by the condition $u'(\tau) = 0$

$$u_m = \left\{\frac{s+1}{2A_0}\left[u_0'^2 + \frac{2A_0}{s+1} u_0^{s+1} - 2\frac{G}{k} u_0' \cos\alpha\right]\right\}^{1/(s+1)} \tag{15}$$

Solving Eq. (14) for $u'(\tau)$ and confining ourselves to quantities of the first order as $k \to \infty$, we obtain a differential equation with separable variables, whose solution is

$$\int_{\nu_0}^{\nu} \frac{d\eta}{\sqrt{1 - \eta^{s+1}}} = \Omega\tau, \quad \nu = \mu^{1/(s+1)}, \quad \nu_0 = \mu_0^{1/(s+1)} \tag{16}$$

This formula is equivalent to (11). The first equality of (12) follows from (11).

In relation to the impact under discussion, we must assume that

$$u_0 = q_0 = 0, \quad u_0' = q_0' = s_0/s_1, \quad E^* = G, \quad A_0 = 1/s_1, \quad \mu_0 = 0, \quad s = {}^3/_2$$

The quantity s depends on the surface curvature at the point of contact [3]. Hence

$$\Omega = \frac{4}{5}\kappa\left(1 + \frac{s_1}{s_0}\frac{\sqrt{2}E^*}{s_3}\right)^{2/s_1}, \quad \kappa = \sqrt{\pi}\frac{\Gamma({}^2/_5)}{\Gamma({}^9/_{10})} \tag{17}$$

Using (12), we obtain

$$P_m = (1 - W_1)^{3/5} P_m^0, \quad t_1 = (1 - W_1)^{-3/5} t_1^0 \tag{18}$$

$$W_1 = \frac{\sqrt{6}}{\pi^6}\frac{1}{1 - \sigma^2}\sqrt{\frac{v}{v_0}}\frac{i^{3/2} R^{1/2}}{r^2}\beta^4$$

where t_1 and t_1^0 are the durations of the impact for a rod and a half-space, respectively, i is the radius of inertia of the rod cross-section relative to the principal axis perpendicular to the velocity of the ball, r is the radius of the rod cross-section $(r=\sqrt{(F/\eta)}$, where F is the cross-sectional area of the rod), R is the radius of the ball, B is the flexibility of the rod $(\beta=l/i$, where l is the length of the rod), v_0 is the velocity of the before impact ball, v is the velocity of longitudinal waves in the rod, and σ is Poisson's ratio (the rod and ball are assumed to be made of the same material).

Note that the function $q=q(\tau)$ is the superposition of high-frequency low-amplitude vibrations on a slow process. This follows from the formula for $q'(\tau)$, derived by solving the quadratic equation (14). The high-frequency vibrations have actually been observed experimentally [4].

Let us assume now that

$$s_3 \ll 1 \tag{19}$$

(the rod has high flexibility and a low-frequency spectrum of natural modes of vibration). An approximate but simple method of solution is to use the first term in the series in $L(p)$ and, in view of (19), replace the sine by its argument. Thus, an equation analogous to the equation of Hertz's impact theory is obtained, so that by the known method of [2] we can find

$$P_m = P_m^0 (1+2m/M_1)^{-3/5}, \quad t_1 = (1+2m/M_1)^{-2/5} t_1^0 \tag{20}$$

where m and M_1 are the mass of the ball and the rod, respectively.

A more accurate result can be obtained if we assume, besides (19), that $s_2\sqrt{s_3} \ll 1$ or, after substituting s_2 and s_3

$$\frac{6}{\pi^{9/4}}\sqrt{\frac{2}{5}}\left(\frac{8}{15}\right)^{1/5}\frac{1}{(1+\sigma^2)^{1/5}}\left(\frac{v}{v_0}\right)^{1/10}\left(\frac{r}{i}\right)^{2/5}\left(\frac{Q}{r^3}\right)^{4/5}\left(\frac{R}{i}\right)^{1/10} \ll 1$$

(where Q is the volume of the impacting body), which is equivalent, for example, to the assumption that the impacting body is small. Substituting the expression for $\theta_2(0|\cdot)$, obtained by the imaginary Jacobi transformation, into the last equation of (3) and integrating by parts in the inner integral, we see that the principal term in the series for $\theta_2(0|\cdot)$, is the one with zero summation index; on our assumption (19), all the other terms are rapidly oscillating exponential functions of an imaginary argument, so that their contribution to the integral with respect to τ_1 is asymptotically small.

Thus, we obtain

$$L(p) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{s_3}\int_0^\tau (\tau-\tau_1)^{1/2} p(\tau_1)\,d\tau_1 \tag{21}$$

Substituting this expression into (1) we obtain an integral equation with the small parameter $s_2\sqrt{s_3}$, which can be handled by linearization of its kernel (the operator on the left of (1) is compact in the Banach space C [4]; similar practical methods of linearization can be found in [5, 6]). Approximating the radical in (21) by a Chebyshev polynomial of the first kind in L^2, we obtain, as before, a solution of Eq. (1), and hence the relations

$$\begin{aligned} &P_m = (1+W_2)P_m^0, \quad t_1 = (1+W_2)^{-2/5} t_1^0 \\ &W_2 = \frac{6}{\pi^2}\sqrt{\frac{2}{5}}\left(\frac{18}{15}\right)^{1/5}\frac{\Gamma(^3/_4)}{\Gamma(^1/_4)\kappa} \times \\ &\times \frac{1}{(1-\sigma^2)^{1/5}}\left(\frac{v}{v_0}\right)^{1/10}\left(\frac{r}{i}\right)^{2/5}\left(\frac{Q}{r^3}\right)^{4/5}\left(\frac{R}{i}\right)^{1/10} \end{aligned} \tag{22}$$

Note that in this case

$$s_3 \ll 1,\ s_2\sqrt{s_3} \ll 1$$

P_m and t_1 are independent of the flexibility of the rod (i.e. of l).

In the intermediate case

$$s_3 \sim 1 \tag{23}$$

confining ourselves to the first term of the series in (1), we approximate the sine by a Chebyshev polynomial of the first kind in L^2 ($\sin s_3(\tau-\tau_1) \sim 2J_1(s_3)(\tau-\tau_1)$, where $J_1(\cdot)$ is the Bessel function of the first kind and first order). Proceeding as before, we obtain

$$\begin{aligned} P_m &= [1+2s_2J_1(s_3)]^{-3/5}P_m^0 \\ t_1 &= [1+2k_2J_1(s_3)]^{-2/5}t_1^0 \end{aligned} \tag{24}$$

To estimate P_m and t_1 in (24), under the condition (23), it is convenient to use the representation [7]

$$J_1(s_3) \sim \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{s_3}}{1+s_3}\cos(s_3+\pi/4)$$

which is asymptotically accurate as $s_3 \to \infty$.

If

$$s_2 \ll 1$$

i.e. if

$$\frac{10}{3\pi^2}\left(\frac{2}{5\pi}\right)^{2/5}\frac{1}{\kappa}\,\frac{1}{(1-\sigma^2)^{2/5}}\left(\frac{v}{v_0}\right)^{1/5}\left(\frac{R}{r}\right)^2\beta \ll 1$$

we can use the averaging method [8]. Averaging the integrand in the second relation of (3) with respect to τ and differentiating (1), we obtain an autonomous differential equation which can be solved in closed form, e.g. by using the energy integral.

We will solve this equation by the linearization method. The substitution

$$p \to Bp^{2/5},\quad q = p^{2/5}$$

where B is a constant depending on the specific approximation used, yields a linear equation whose solution is

$$\begin{aligned} &q = k_0/k_1\,\omega^{-1}\sin\omega\tau,\quad \omega = 2\kappa\sqrt{B/s}(1+W_3) \\ &W_3 = \frac{\pi^4\kappa^2B}{120}\,\frac{s_2}{s_3} \end{aligned} \tag{25}$$

Hence

$$\begin{aligned} P_m &= [{}^5/_4B(1+W_3)]^{-3/4}\cdot P_m^0 \\ t_1 &= (\pi/2)\sqrt{5/B}\,\kappa^{-1}\sqrt{1+W_3}\,t_1^0 \\ \frac{s_2}{s_3} &= \frac{5}{3}\sqrt{\frac{5\pi}{2}}\,\frac{1}{\pi^2\kappa^2}\,\frac{1}{(1+\sigma^2)^{4/5}}\,\frac{Ri}{r^2}\beta^3 \end{aligned} \tag{26}$$

We shall examine the accuracy of these results in the special case of a half-space, where the solution is known. Since in a semibounded body $W_1 = W_2 = W_3 = s_2 = 0$, relations (18), (20), (22) and (25) become exact. Formulae (26), obtained by linearization in the case of a semibounded body, may be written in the form

$$\eta_p = \left(\frac{4}{5B}\right)^{3/4}, \quad \eta_t = \sqrt{\frac{5}{3}}\,\frac{\pi}{2\kappa} \tag{27}$$

where η_p and η_t are the ratios of the numbers P_m and t_1 for a semibounded body, obtained from the solution (25) of the linearized equation, to their exact values P_m^0 and t_1^0; the relative errors of linearization are $\Delta p = |1-\eta_p|$, $\Delta t = |1-\eta_t|$. The function $p^{2/3}$ was linearized for the following approximation methods: Chebyshev approximation in L^∞; approximation of the first and second kind in L^2; Legendre approximation; linear interpolation at points 0,1. The constant B was then determined so that $p \sim Bp^{2/3}$. The results of the linearization are tabulated below

B	0.855	0.889	0.900	0.934	1.00
η_p	0.952	0.925	0.920	0.889	0.847
η_t	1.030	1.010	1.005	0.991	0.954
$\Delta p \times 10^2$	5	7.5	8	11	15
$\Delta t \times 10^2$	3	1	0.5	1	5

The minimum error $\Delta t = 0.5\%$ was achieved in L^∞ (a linearization error of the same order (0.23%) was obtained [9] in determining the frequency of non-linear vibrations). The largest error was obtained in interpolation (which makes the approximation at isolated points and not over an entire interval, as in the other methods [10, 11]). All the errors Δp are greater than the corresponding Δts, owing to the improved accuracy of the approximation at the point $\tau = 0$ (and at the conjugate point $\tau = 1$ in the autonomous ball–rod system), while the quantity P_m is determined at $\tau = 1/2$. The minimum quantity corresponds to Chebyshev approximation of the second kind, which yields high accuracy at $\tau \sim 1/2$ [10, 11].

The methods discussed can be used to investigate lateral impact in more-complicated systems: beams, plates, shells, etc. In those cases, by representing the deformation due to various loads by eigenfunction expansions [12] one can, using Timoshenko's method [1], derive integral equations of type (1), whose solutions can be investigated by the above methods. Relations (18), (20), (22), (24) and (26) will then have the same form.

More-complex models of deformation, such as rheological models, may be investigated similarly; in linear viscoelastic systems the correspondence principle can be used to determine the one-sided Fourier transforms of the solutions, to introduce complex constants of elasticity, and then to invert the transforms.

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